The Linear Algebra of Handle Attachments

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Following [Kup, §16], we just want to get an idea of the algebra. Every compact manifold can be built by successively attaching handles to $\emptyset$. Consider then a manifold with handle decomposition

$$W = \partial(W \times I) + \sum(\varphi_i)$$

We may assume by handle rearrangment lemmas that the order of the index of the φ_i is increasing. By retracting a handle attachment to its core we can use the handle decomposition to produce a CW decomposition of the manifold that is *homotopy equivalent* to the original manfold, call it X. Moreover we have that the k-cells are in bijection with the k-handles (this is a good way to think of the index it is the dimension of the core).

If W_k is the k-th step in the handle decomposition $\partial(W \times I) + \sum_{i \leq k}(\varphi_i)$ and X_k is the k-cells of the CW decomposition then we also have

$$H_*(W_k, W_{k-1}) \cong H_*(X_k, X_{k-1})$$

thus we can compute the homology of W by computing the cellular homology of X or vice versa. By composing the maps in the LES of homology we have the map

$$\partial_* : H_*(X_k, X_{k-1}) \to H_{*-1}(X_{k-1}, X_{k-2})$$

If we take $\mathbb{Q}$ coefficients then this is a linear map between vector spaces and is therefore represented by a matrix. The claim is that we can compute the i, jth entry in the following way: This entry represents the degree of the attaching map of some k cell to some $k-1$ cell, by above these two cells correspond to two handles, and instead of the degree of the attaching map one can look at the intersection number of the attaching sphere of the k handle and the transverse sphere of the $k-1$ handle.

Remark. Poincare duality as interpreted on homology interchanges attaching spheres and transverse spheres!

Now we have a matrix and some operations on the handle decompostion (rearrangment and cancelation etc) and we should see what these geometric operations induce on the level of the linear algebra. First we can deal with either rows or columns, its just changing a basis, we will fix rows for convenience. We can *multiply a row by* -1, this corresponds to the arbitrary choice of orientation on the cells or handle attachments. We can *interchange two rows* which is just relabeling any two handles. We can also *add one row to another*, this is [Kos93, VIII. Thm 1.3] but geometrically [Kos93, VIII Lem 1.1], the idea is to take two disjoint embedded spheres S_1, S_2 and isotope S_1 to being S_1 connected to a copy of S_2 by a tube. When these embeddings are then used to attach discs we get the required row operation. Finally we may *remove a row and column pair with a one in their intersection and zeroes everywhere else*, this is our handle cancelation lemma.

Remark. Thus all the elementary row operations over the ring of coefficients for the homology can be performed, and row reduction can be interpreted as rearanging the handle decomposition into a simpler form.

Remark. This is a good reminded that we can similarly interpret these row operations in terms of CW moves, changing orientation, relabeling cells, deforming one cell into another and twisting it over.

References

[Kos93] Antoni A. Kosinski. *Differential manifolds.* Number v. 138 in Pure and applied mathematics. Academic Press, Boston, 1993.

[Kup] Alexander Kupers. Lectures on diffeomorphism groups of manifolds, version February 22, 2019.